ENGINEERING ECONOMY

Engineering Economic Decisions
Foundations of Engineering Economy
Factors: Effect of Time and Interest on Money
Engineering Economy

**Description:**
Economic analysis for engineering decision making; The finance function in an industrial enterprise, time value of money; Basic interest formulas; Annual cost comparison; Present value analysis; Rate of return; Depreciation and taxes; Multiple alternatives; Mathematical models for equipment replacement; Decision analysis; Concepts of cost engineering.
Why Engineering Economy is Important to Engineers

- Engineers design and create
- Designing involves economic decisions
- Engineers must be able to incorporate economic analysis into their creative efforts
- Often engineers must select and implement from multiple alternatives
- Understanding and applying time value of money, economic equivalence, and cost estimation are vital for engineers
- A proper economic analysis for selection and execution is a fundamental task of engineering
Chapter Opening Story 1 - Bose Corporation

- Dr. Amar Bose, a graduate of electrical engineering, an MIT professor, and Chairman of Bose Corporation.
- He invented a directional home speaker system that reproduces the concert experience.
- He formed Bose Corporation in 1964 and became the world’s No.1 speaker maker.
- He became the 288th wealthiest American in 2002 by Forbes magazine.
Opening Story 2: Google
How Google Works

1. The Web server sends the query to the index servers—it tells which pages contain the words that match the query.

2. The query travels to the Doc servers (which retrieve the stored documents) and snippets are generated to describe each search result.

3. The search results are returned to the user in a fraction of a second.
A Little Google History

• 1995
  • Developed in dorm room by Larry Page and Sergey Brin, graduate students at Stanford University
  • Nicknamed BackRub (reflecting great taste… 😊)

• 1998
  • Raised $25 million to set up Google, Inc.
  • Ran 100,000 queries a day out of a garage in Menlo Park

• 2005
  • Over 4,000 employees worldwide
  • Over 8 billion pages indexed
  • Estimated market value over $100 billion
  • As of today, the value of Google is likely to be in the hundreds of billions range
Engineering Economics Overview

- Rational Decision-Making Process
- Economic Decisions
- Predicting Future
- Role of Engineers in Business
- Large-scale engineering projects
- Types of strategic engineering economic decisions
Rational Decision-Making Process

1. Recognize a decision problem
2. Define the goals or objectives
3. Collect all the relevant information
4. Identify a set of feasible decision alternatives
5. Select the decision criterion to use
6. Select the best alternative
A Simple Illustrative Example: Car to Lease – Saturn or Honda?

- Recognize the decision problem
- Collect all needed (relevant) information
- Identify the set of feasible decision alternatives
- Define the key objectives and constraints
- Select the best possible and implementable decision alternative

- Need to lease a car
- Gather technical and financial data
- Select cars to consider
- Wanted: small cash outlay, safety, good performance, aesthetics,…
- Choice between Saturn and Honda (or others)
- Select a car (i.e., Honda, Saturn or another brand)
# Financial Data Required to Make an Economic Decision

## Table 1.1: Financial Data for Auto Leasing: Saturn versus Honda

<table>
<thead>
<tr>
<th>Auto Leasing</th>
<th>Saturn</th>
<th>Honda</th>
<th>Difference Saturn – Honda</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Manufacturer’s suggested retail price (MSRP)</td>
<td>$15,573</td>
<td>$15,810</td>
<td>$-273</td>
</tr>
<tr>
<td>2. Lease length</td>
<td>48 months</td>
<td>48 months</td>
<td></td>
</tr>
<tr>
<td>3. Allowed mileage</td>
<td>48,000 miles</td>
<td>48,000 miles</td>
<td></td>
</tr>
<tr>
<td>4. Monthly lease payment</td>
<td>$219</td>
<td>$248</td>
<td>$-29</td>
</tr>
<tr>
<td>5. Mileage surcharge over 36,000 miles</td>
<td>$0.20 per mile</td>
<td>$0.15 per mile</td>
<td>$+0.05 per mile</td>
</tr>
<tr>
<td>6. Disposition fee at lease end</td>
<td>$0</td>
<td>$250</td>
<td>$250</td>
</tr>
<tr>
<td>7. Total due at signing:</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- First month’s lease payment | $219 | $248 |
- Down payment | $1,100 | $800 |
- Administrative fee | $495 | $0 |
- Refundable security deposit | $200 | $225 |

Total | $2,014 | $1,273 | $+741

*Models compared: The 2003 Saturn ION3 with automatic transmission and A/C and the 2003 Honda Civic DX coupe with automatic transmission and A/C.*

*Disposition fee: This is a paperwork charge for getting the vehicle ready for resale after the lease end.*
Engineering Economic Decisions

Planning → Investment → Marketing → Manufacturing → Profit

- EMR 301 – ENGINEERING ECONOMY
Predicting the Future

- Estimating a Required investment
- Forecasting a product demand
- Estimating a selling price
- Estimating a manufacturing cost
- Estimating a product life and the profitability of continuing production
Role of Engineers in Business

Create & Design

- Engineering Projects

Analyze

- Production Methods
- Engineering Safety
- Environmental Impacts
- Market Assessment

Evaluate

- Expected Profitability
- Timing of Cash Flows
- Degree of Financial Risk

Evaluate

- Impact on Financial Statements
- Firm’s Market Value
- Stock Price
Accounting Vs. Accounting

- **Accounting**
  - Evaluating past performance
  - Past

- **Engineering Economy**
  - Evaluating and predicting future events
  - Present
  - Future
Two Factors in Engineering Economic Decisions

Objectives, available resources, time and uncertainty are the key defining aspects of all engineering economic decisions.

The factors of **time** and **uncertainty** are the defining aspects of any engineering economic decisions.
A Large-Scale Engineering Project

- Requires **a large sum of investment**
- Takes a **long time** to see the financial outcomes
- **Difficult to predict** the revenue and cost streams
- **Can be very risky**
Types of Strategic Engineering Economic Decisions in Manufacturing Sector

- Service Improvement
- Equipment and Process Selection
- Equipment Replacement
- New Product and Product Expansion
- Cost Reduction
- Profit Maximization

Engineers have to make decisions under resource constraints, and in presence of uncertainty.
Types of Strategic Engineering Economic Decisions in Service Sector

- Commercial Transportation
- Logistics and Distribution
- Healthcare Industry
- Electronic Markets and Auctions
- Financial Engineering
- Retails
- Hospitality and Entertainment
- Customer Service and Maintenance
Equation & Process Selection

Description | Plastic SMC | Steel Sheet Stock
--- | --- | ---
Material cost ($/kg) | $1.65 | $0.77
Machinery investment | $2.1 million | $24.2 million
Tooling investment | $0.683 million | $4 million
Cycle time (minute/part) | 2.0 | 0.1
Equipment Replacement Problem

• **Now** is the time to replace the old machine?
• If not, **when** is the right time to replace the old equipment?
New Product and Product Expansion

• Shall we build or acquire a new facility to meet the increased demand?
• Is it worth spending money to market a new product?
Example - MACH 3 Project

- **R&D investment**: $750 million
- **Product promotion through advertising**: $300 million
- **Priced to sell at 35% higher than Sensor Excel** (about $1.50 extra per shave).

**Question 1**: Would consumers pay $1.50 extra for a shave with greater smoothness and less irritation?

**Question 2**: What would happen if the blade consumption dropped more than 10% due to the longer blade life of the new razor?
Cost Reduction

• Should a company buy equipment to perform an operation now done manually?
• Should spend money now in order to save more money later?
Fundamental Principles of Engineering Economics

- **Principle 1**: An instant (nearby) dollar is worth more than a distant dollar
- **Principle 2**: All it counts is the differences among alternatives
- **Principle 3**: Marginal revenue must exceed marginal cost
- **Principle 4**: Additional risk is not taken without the expected additional return
Principle 1: A nearby dollar is worth more than a distant dollar

Today

6-month later
Principle 2: All it counts is the differences among alternatives

<table>
<thead>
<tr>
<th>Option</th>
<th>Monthly Fuel Cost</th>
<th>Monthly Maintenance</th>
<th>Cash outlay at signing</th>
<th>Monthly payment</th>
<th>Salvage Value at end of year 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy</td>
<td>$960</td>
<td>$550</td>
<td>$6,500</td>
<td>$350</td>
<td>$9,000</td>
</tr>
<tr>
<td>Lease</td>
<td>$960</td>
<td>$550</td>
<td>$2,400</td>
<td>$550</td>
<td>0</td>
</tr>
</tbody>
</table>

Irrelevant items in decision making
Principle 3: Marginal revenue must exceed marginal cost
**Principle 4:** Additional risk is not taken without the expected additional return

<table>
<thead>
<tr>
<th>Investment Class</th>
<th>Potential Risk</th>
<th>Expected Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>Savings account (cash)</td>
<td>Low/None</td>
<td>1.5%</td>
</tr>
<tr>
<td>Bond (debt)</td>
<td>Moderate</td>
<td>4.8%</td>
</tr>
<tr>
<td>Stock (equity)</td>
<td>High</td>
<td>11.5%</td>
</tr>
</tbody>
</table>
Summary

• The term engineering economic decision refers to all investment decisions relating to engineering projects.

• The five main types of engineering economic decisions are
  • (1) service improvement,
  • (2) equipment and process selection,
  • (3) equipment replacement,
  • (4) new product and product expansion, and
  • (5) cost reduction.

• The factors of time and uncertainty are the defining aspects of any investment project.
Time Value of Money (TVM)

Description: TVM explains the change in the amount of money over time for funds owed by or owned by a corporation (or individual)

- Corporate investments are expected to earn a return
- Investment involves money
- Money has a ‘time value’

The time value of money is the most important concept in engineering economy
Engineering Economy

- Engineering Economy involves
  - Formulating
  - Estimating, and
  - Evaluating
    expected economic outcomes of alternatives designed to accomplish a defined purpose

- Easy-to-use math techniques simplify the evaluation
- Estimates of economic outcomes can be deterministic or stochastic in nature
General Steps for Decision Making Processes

1. Understand the problem – define objectives
2. Collect relevant information
3. Define the set of feasible alternatives
4. Identify the criteria for decision making
5. Evaluate the alternatives and apply sensitivity analysis
6. Select the “best” alternative
7. Implement the alternative and monitor results
Steps in an Engineering Economy Study

1. Problem description | Objective statement
   - Available data
   - Alternatives for solution

2. Cash flows and other estimates
   - One or more approaches to meet objective
   - Expected life
   - Revenues
   - Costs
   - Taxes
   - Project financing

3. Measure of worth criterion
   - PW, ROR, B/C, etc.

4. Engineering economic analysis
   - Consider:
     - Noneconomic factors
     - Sensitivity analysis
     - Risk analysis

5. Best alternative selection

6. Implementation and monitoring
   - Time passes

7. New problem description
   - New engineering economy study begins
Ethics – Different Levels

- Universal morals or ethics – Fundamental beliefs: stealing, lying, harming or murdering another are wrong
- Personal morals or ethics – Beliefs that an individual has and maintains over time; how a universal moral is interpreted and used by each person
- Professional or engineering ethics – Formal standard or code that guides a person in work activities and decision making
Code of Ethics for Engineers

All disciplines have a formal code of ethics. National Society of Professional Engineers (NSPE) maintains a code specifically for engineers; many engineering professional societies have their own code.
Interest and Interest Rate (borrower’s perspective - paid)

- **Interest** – the manifestation of the time value of money
  - Fee that one pays to use someone else’s money
  - Difference between an ending amount of money and a beginning amount of money

  \[ \text{Interest} = \text{amount owed now} - \text{principal (initial amount)} \]

- **Interest rate** – Interest paid over a time period expressed as a percentage of principal

  Time unit = interest period (if no interest period is stated, a 1 year interest rate is assumed)

\[
\text{Interest rate (\%) } = \frac{\text{interest accrued per time unit}}{\text{principal}} \times 100\%
\]
Interest and Interest Rate (borrower’s perspective - paid)

• Example 1

• Example 2
Rate of Return – ROR / Return on Investment -ROI (saver / lender / investor’s perspective - earned)

- Interest earned over a period of time is expressed as a percentage of the original amount (principal)

\[
\text{Rate of return (\%)} = \frac{\text{interest accrued per time unit}}{\text{original amount}} \times 100\%
\]

- Borrower’s perspective – interest rate paid
- Lender’s or investor’s perspective – rate of return earned
Rate of Return – ROR / Return on Investment - ROI (saver / lender / investor’s perspective - earned)

- Example 3
Interest paid

Interest rate

Interest earned

Rate of return
Commonly used Symbols

\[ t = \text{time, usually in periods such as years or months} \]
\[ P = \text{value or amount of money at a time } t \]
\[ \text{designated as present or time 0} \]
\[ F = \text{value or amount of money at some future time, such as at } t = n \]
\[ \text{periods in the future} \]
\[ A = \text{series of consecutive, equal, end-of-period amounts of money} \]
\[ n = \text{number of interest periods; years, months} \]
\[ i = \text{interest rate or rate of return per time period; percent per year or month} \]
INFLATION

• Inflation: Changing value of money that is forced upon a country’s currency by inflation.

Cost and revenue cash flow estimates increase over time.

Inflation contributes:
• A reduction in purchasing power
• An increase in the CPI (consumer price index)
• An increase in the cost of equipment and its maintenance
• An increase in the cost salaried professionals and hourly employees
• A reduction in the real rate of return on personal savings and corporate investments
Cash Flows: Terms

• **Cash Inflows** – Revenues (R), receipts, incomes, savings generated by projects and activities that flow in. Plus sign used

• **Cash Outflows** – Disbursements (D), costs, expenses, taxes caused by projects and activities that flow out. Minus sign used

• Net Cash Flow (NCF) for each time period:
  \[
  \text{NCF} = \text{cash inflows} - \text{cash outflows} = R - D
  \]

• End-of-period assumption:
  Funds flow at the end of a given interest period
Cash Flows: Estimating

✓ Point estimate – A single-value estimate of a cash flow element of an alternative
  
  Cash inflow: Income = $150,000 per month

✓ Range estimate – Min and max values that estimate the cash flow
  
  Cash outflow: Cost is between $2.5 M and $3.2 M

  Point estimates are commonly used; however, range estimates with probabilities attached provide a better understanding of variability of economic parameters used to make decisions
Cash Flow Diagrams

What a typical cash flow diagram might look like

Draw a time line

Always assume end-of-period cash flows

Show the cash flows (to approximate scale)

Cash flows are shown as directed arrows: + (up) for inflow
- (down) for outflow
Cash Flow Diagram Example

Plot observed cash flows over last 8 years and estimated sale next year for $150. Show present worth (P) arrow at present time, t = 0

<table>
<thead>
<tr>
<th>End of Year</th>
<th>Income</th>
<th>Cost</th>
<th>Net Cash Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>−7</td>
<td>$ 0</td>
<td>$2500</td>
<td>$−2500</td>
</tr>
<tr>
<td>−6</td>
<td>750</td>
<td>100</td>
<td>650</td>
</tr>
<tr>
<td>−5</td>
<td>750</td>
<td>125</td>
<td>625</td>
</tr>
<tr>
<td>−4</td>
<td>750</td>
<td>150</td>
<td>600</td>
</tr>
<tr>
<td>−3</td>
<td>750</td>
<td>175</td>
<td>575</td>
</tr>
<tr>
<td>−2</td>
<td>750</td>
<td>200</td>
<td>550</td>
</tr>
<tr>
<td>−1</td>
<td>750</td>
<td>225</td>
<td>525</td>
</tr>
<tr>
<td>0</td>
<td>750</td>
<td>250</td>
<td>500</td>
</tr>
<tr>
<td>1</td>
<td>750 + 150</td>
<td>275</td>
<td>625</td>
</tr>
</tbody>
</table>

$650  \rightarrow \rightarrow \rightarrow \rightarrow P = ? \rightarrow \rightarrow \rightarrow \rightarrow $625

$2500 \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow$
Example 4
Economic Equivalence

Definition: Combination of interest rate (rate of return) and time value of money to determine different amounts of money at different points in time that are economically equivalent.

How it works: Use rate $i$ and time $t$ in upcoming relations to move money (values of $P$, $F$ and $A$) between time points $t = 0, 1, \ldots, n$ to make them equivalent (not equal) at the rate $i$. 


Example of Equivalence

Different sums of money at different times may be equal in economic value at a given rate.

$100 now is economically equivalent to $110 one year from now, if the $100 is invested at a rate of 10% per year.
Simple and Compound Interest

- **Simple Interest**
  
  Interest is calculated using principal only
  
  \[ I = (\text{principal})(\text{number of periods})(\text{interest rate}) \]
  
  \[ I = P \cdot n \cdot i \]

**Example:** $100,000 lent for 3 years at simple \( i = 10\% \) per year. What is repayment after 3 years?

\[
\text{Interest} = 100,000(3)(0.10) = \$30,000
\]

\[
\text{Total due} = 100,000 + 30,000 = \$130,000
\]
Simple Interest

Example 5
• **Compound Interest**

  Interest is based on principal plus all accrued interest.

  That is, interest compounds over time.

  
  \[ \text{Interest} = (\text{principal} + \text{all accrued interest}) \times \text{(interest rate)} \]

  
  Interest for time period \( t \) is

  \[ I_t = \left( P + \sum_{j=1}^{t-1} I_j \right) \times (i) \]
Compound Interest Example

Example: $100,000 lent for 3 years at $i = 10\%$ per year compounded. What is repayment after 3 years?

<table>
<thead>
<tr>
<th>Year</th>
<th>Interest</th>
<th>Total Due</th>
</tr>
</thead>
<tbody>
<tr>
<td>Year 1</td>
<td>$10,000</td>
<td>$110,000</td>
</tr>
<tr>
<td>Year 2</td>
<td>$11,000</td>
<td>$121,000</td>
</tr>
<tr>
<td>Year 3</td>
<td>$12,100</td>
<td>$133,100</td>
</tr>
</tbody>
</table>

Compounded: $133,100  
Simple: $130,000

Practical Solution for the amount due the stated time in the future

\[ F = P \times (1 + i)^n \]

$133,100 = 100,000 \times (1.10)^3$
Compound Interest

Example 6
Minimum Attractive Rate of Return (MARR)

- MARR is a reasonable rate of return (percent) established for evaluating and selecting alternatives.
- An investment is justified economically if it is expected to return at least the MARR.
- Also termed *hurdle rate*, *benchmark rate* and *cutoff rate*.
MARR Characteristics

• MARR is established by the financial managers of the firm
• MARR is fundamentally connected to the cost of capital
• Both types of capital financing are used to determine the weighted average cost of capital (WACC) and the MARR
• MARR usually considers the risk inherent to a project
Types of Financing

In general, capital is developed in two ways;

- **Equity Financing** – Funds either from retained earnings, new stock issues, or owner’s infusion of money.

- **Debt Financing** – Borrowed funds from outside sources – loans, bonds, mortgages, venture capital pools, etc. Interest is paid to the lender on these funds.

For an economically justified project:

\[ \text{ROR} \geq \text{MARR} > \text{WACC (cost of capital)} \]
Opportunity Cost

- Definition: Largest rate of return of all projects not accepted (forgone) due to a lack of capital funds.
- If no MARR is set, the ROR of the first project not undertaken establishes the opportunity cost.

- The loss of potential gain from other alternatives when one alternative is chosen.

**Example:** Assume MARR = 10%. Project A, not funded due to lack of funds, is projected to have ROR$_A$ = 13%. Project B has ROR$_B$ = 15% and is funded because it costs less than A.

Opportunity cost is 13%, i.e., the opportunity to make an additional 13% is forgone by not funding project A.
Rule of 72

- A common question most often asked by investors is: how long will it take for my investment to double in value
  - Must have a known or assumed compound interest rate in advance
  - Assume a rate of 13%/year to illustrate....

The Rule of 72’s for Interest

- The approximate time for an investment to double in value given the compound interest rate is:
  - Estimated time (n) = 72 / i
  - For i = 13% = 5.54 years
Rule of 72

- The Rule of 72’s for Interest
  - Likewise one can estimate the required interest rate for an investment to double in value over time as:

  \[ i_{\text{approximate}} = \frac{72}{n} \]

  - Assume we want an investment to double in say 3 years.
  - Estimate \( i = \) rate would be: \( \frac{72}{3} = 24\% \)
Chapter Summary

• Engineering Economy fundamentals
  ❖ Time value of money
  ❖ Economic equivalence
  ❖ Introduction to capital funding and MARR
  ❖ Spreadsheet functions
• Interest rate and rate of return
  ❖ Simple and compound interest
• Cash flow estimation
  ❖ Cash flow diagrams
  ❖ End-of-period assumption
  ❖ Net cash flow
  ❖ Perspectives taken for cash flow estimation
• Ethics
  ❖ Universal morals and personal morals
  ❖ Professional and engineering ethics (Code of Ethics)
ENGINEERING ECONOMY

Factors: How Time and Interest Affect Money
LEARNING OUTCOMES

1. F/P and P/F Factors
2. P/A and A/P Factors
3. F/A and A/F Factors
4. Factor Values
5. Arithmetic Gradient
6. Geometric Gradient
7. Find i or n
Single Payment Factors (F/P and P/F)

Single payment factors involve only $P$ and $F$. Cash flow diagrams are as follows:

Formulas are as follows:

$$F = P(1 + i)^n$$
$$P = F \left[ \frac{1}{(1 + i)^n} \right]$$

Terms in parentheses or brackets are called factors. Values are in tables for $i$ and $n$ values.

Factors are represented in standard factor notation such as $(F/P, i, n)$, where letter to left of slash is what is sought; letter to right represents what is given.
F/P and P/F for Spreadsheets

Future value F is calculated using FV function:

\[ = FV(i\%, n, , P) \]

Present value P is calculated using PV function:

\[ = PV(i\%, n, , F) \]

Note the use of double commas in each function.
Example: Finding Future Value

A person deposits $5000 into an account which pays interest at a rate of 8% per year. The amount in the account after 10 years is closest to:

(A) $2,792  (B) $9,000  (C) $10,795  (D) $12,165

The cash flow diagram is:

Solution:

\[ F = P(F/P, i, n) \]

\[ = 5000(F/P, 8\%, 10) \]

\[ = 5000(2.1589) \]

\[ = $10,794.50 \]
Finding Future Value

Example 6
Example: Finding Present Value

A small company wants to make a single deposit now so it will have enough money to purchase a backhoe costing $50,000 five years from now. If the account will earn interest of 10% per year, the amount that must be deposited now is nearest to:

(A) $10,000  (B) $31,050  (C) $33,250  (D) $319,160

The cash flow diagram is:

Solution:

\[ P = F(P/F, i, n) \]
\[ = 50,000(0.6209) \]
\[ = 31,045 \]
Finding Present Value

Example 7
Finding Present/Future Value

Example 8
Uniform Series Involving P/A and A/P

The uniform series factors that involve P and A are derived as follows:

(1) Cash flow occurs in **consecutive** interest periods
(2) Cash flow amount is **same** in each interest period

The cash flow diagrams are:

\[
P = \text{?} \]

\[
A = \text{Given} \\
0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5
\]

\[
P = A(P/A, i, n) \quad \text{Standard Factor Notation} \quad A = P(A/P, i, n)
\]

**Note:** P is one period **Ahead** of first A value
Example: Uniform Series Involving P/A

A chemical engineer believes that by modifying the structure of a certain water treatment polymer, his company would earn an extra $5000 per year. At an interest rate of 10% per year, how much could the company afford to spend now to just break even over a 5 year project period?

(A) $11,170   (B) 13,640   (C) $15,300   (D) $18,950

Solution:

\[ P = 5000(P/A, 10\%, 5) \]
\[ = 5000(3.7908) \]
\[ = $18,954 \]
Uniform Series Involving P/A

Example 9
Uniform Series Involving F/A and A/F

The uniform series factors that involve $F$ and $A$ are derived as follows:

1. Cash flow occurs in *consecutive* interest periods
2. Last cash flow occurs in *same* period as $F$

Cash flow diagrams are:

$F = A(F/A, i, n) \quad \text{Standard Factor Notation} \quad A = F(A/F, i, n)$

**Note:** $F$ takes place in the *same* period as last $A$
Uniform Series Involving F/A

Example 10

Example 11
Arithmetic Gradients

Arithmetic gradients change by the same amount each period.

The cash flow diagram for the $P_G$ of an arithmetic gradient is:

G starts between periods 1 and 2 (not between 0 and 1)

This is because cash flow in year 1 is usually not equal to G and is handled separately as a base amount.

Note that $P_G$ is located Two Periods Ahead of the first change that is equal to G.

Standard factor notation is $P_G = G(P/G,i,n)$.
Typical Arithmetic Gradient Cash Flow

\[ P_T = P_A + P_G = 400(P/A, 10\%, 5) + 50(P/G, 10\%, 5) \]

\( P_T = ? \)

\( i = 10\% \)

Amount in year 1 is base amount

This diagram = this base amount plus this gradient

\( P_A = 400(P/A, 10\%, 5) \)

\( P_G = 50(P/G, 10\%, 5) \)

2-14
Converting Arithmetic Gradients to P

Equation: Arithmetic Gradient Present Worth Factor

\[ P = G(P/G, i, n) \]

& Figure P 68 2-13
Arithmetic Gradients

Example 12
Converting Arithmetic Gradient to A

Arithmetic gradient can be converted into equivalent A value using \(G(A/G, i, n)\)

General equation when base amount is involved is

\[ A = \text{base amount} + G(A/G, i, n) \]

For decreasing gradients, change plus sign to minus

\[ A = \text{base amount} - G(A/G, i, n) \]
Converting Arithmetic Gradients to A

Equation : Arithmetic Gradient Uniform Series Factor
A = (A/G, i, n)

& Figure P 68 2-14
Example: Arithmetic Gradient

The present worth of $400 in year 1 and amounts increasing by $30 per year through year 5 at an interest rate of 12% per year is closest to:

(A) $1532  (B) $1,634  (C) $1,744  (D) $1,829

Solution:

\[ P_T = 400(P/A, 12\%, 5) + 30(P/G, 12\%, 5) \]
\[ = 400(3.6048) + 30(6.3970) \]
\[ = $1,633.83 \]

Answer is (B)

The cash flow could also be converted into an \( A \) value as follows:

\[ A = 400 + 30(A/G, 12\%, 5) \]
\[ = 400 + 30(1.7746) \]
\[ = $453.24 \]
Arithmetic Gradients

Example 13
Geometric Gradients

Geometric gradients change by the same percentage each period

Cash flow diagram for present worth of geometric gradient

\[ P_g = ? \]

1 2 3 4 n

0 \[ A_1 \]

\[ A_1(1+g)^1 \]

\[ A_1(1+g)^2 \]

\[ A_1(1+g)^{n-1} \]

Note: \( g \) starts between periods 1 and 2

There are no tables for geometric factors

Use following equation for \( g \neq i \):

\[ P_g = A_1 \left\{1 - \left(\frac{1+g}{1+i}\right)^n\right\}/(i-g)\]

where: \( A_1 = \text{cash flow in period 1} \)

\( g = \text{rate of increase} \)

If \( g = i \), \( P_g = A_1n/(1+i) \)

Note: If \( g \) is negative, change signs in front of both \( g \) values
Example: Geometric Gradient

Find the present worth of $1,000 in year 1 and amounts increasing by 7% per year through year 10. Use an interest rate of 12% per year.

(a) $5,670  (b) $7,333  (c) $12,670  (d) $13,550

Solution:

\[ P_g = 1000 \left[ 1 - \left( \frac{1+0.07}{1+0.12} \right)^{10} \right] / (0.12 - 0.07) \]

\[ = $7,333 \]

Answer is (b)

To find \( A \), multiply \( P_g \) by \( (A/P, 12\%, 10) \)
Geometric Gradients

Example 14
Unknown Interest Rate $i$

Unknown interest rate problems involve solving for $i$, given $n$ and 2 other values ($P$, $F$, or $A$)

*(Usually requires a trial and error solution or interpolation in interest tables)*

**Procedure:** Set up equation with all symbols involved and solve for $i$

A contractor purchased equipment for $60,000 which provided income of $16,000 per year for 10 years. The annual rate of return of the investment was closest to:

(a) 15%    (b) 18%    (c) 20%    (d) 23%

**Solution:** Can use either the $P/A$ or $A/P$ factor. Using $A/P$:

$$60,000(A/P, i\%, 10) = 16,000$$

$$A/P, i\%, 10 = 0.26667$$

From $A/P$ column at $n = 10$ in the interest tables, $i$ is between 22% and 24%    

*Answer is (d)*
Example 15
Unknown Recovery Period \( n \)

**Unknown recovery period problems involve solving for \( n \), given \( i \) and 2 other values (\( P, F, \) or \( A \))**

(Like interest rate problems, they usually require a trial & error solution or interpolation in interest tables)

**Procedure:** Set up equation with all symbols involved and solve for \( n \)

A contractor purchased equipment for $60,000 that provided income of $8,000 per year. At an interest rate of 10% per year, the length of time required to recover the investment was closest to:

(a) 10 years  (b) 12 years  (c) 15 years  (d) 18 years

**Solution:** Can use either the \( P/A \) or \( A/P \) factor. Using \( A/P \):

\[
60,000(A/P,10\%,n) = 8,000
\]

\[
(A/P,10\%,n) = 0.13333
\]

From \( A/P \) column in \( i = 10\% \) interest tables, \( n \) is between 14 and 15 years  **Answer is (c)**
Example 16
Summary of Important Points

+ In P/A and A/P factors, P is *one period ahead* of first A
+ In F/A and A/F factors, F is in *same period as last A*
+ To find untabulated factor values, best way is to use formula or spreadsheet
+ For arithmetic gradients, gradient G starts between *periods 1 and 2*
+ Arithmetic gradients have 2 parts, *base amount* (year 1) and *gradient amount*
+ For geometric gradients, gradient g starts between *periods 1 and 2*
+ In geometric gradient formula, $A_1$ is amount in *period 1*
+ To find unknown i or n, *set up equation involving all terms* and solve for i or n
ENGINEERING ECONOMY

Combining Factors
LEARNING OUTCOMES

1. Shifted uniform series
2. Shifted series and single cash flows
3. Shifted gradients
Shifted Uniform Series

A shifted uniform series starts at a time *other than period 1*

The cash flow diagram below is an example of a shifted series

Series starts in period 2, not period 1

Shifted series usually require the use of *multiple factors*

Remember: When using P/A or A/P factor, \( P_A \) is always *one year ahead* of first A

When using F/A or A/F factor, \( F_A \) is in *same year as last A*
Example Using P/A Factor: Shifted Uniform Series

The present worth of the cash flow shown below at \( i = 10\% \) is:

(a) $25,304  
(b) $29,562  
(c) $34,462  
(d) $37,908

**Solution:**

1. Use P/A factor with \( n = 5 \) (for 5 arrows) to get \( P_1 \) in year 1
2. Use P/F factor with \( n = 1 \) to move \( P_1 \) back for \( P_0 \) in year 0

\[
P_0 = P_1 \frac{P/F, 10\%, 1}{P/F, 10\%, 1} = A \left( \frac{P/A, 10\%, 5}{P/F, 10\%, 1} \right) = 10,000 \times 3.7908 \times 0.9091 = $34,462
\]

Answer is (c)
Example 17
How much money would be available in year 10 if $8000 is deposited each year in years 3 through 10 at an interest rate of 10% per year?

Cash flow diagram is:

\[ F_A = 8000(F/A, 10\%, 8) \]

\[ = 8000(11.4359) \]

\[ = $91,487 \]

Solution: Re-number diagram to determine \( n = 8 \) (number of arrows)
Example 18
Shifted Series and Random Single Amounts

For cash flows that include uniform series and randomly placed single amounts:

- Uniform series procedures are applied to the series amounts
- Single amount formulas are applied to the one-time cash flows

The resulting values are then combined per the problem statement

The following slides illustrate the procedure
Example: Series and Random Single Amounts

Find the present worth in year 0 for the cash flows shown using an interest rate of 10% per year.

Solution:

First, re-number cash flow diagram to get n for uniform series: \( n = 8 \)
Example: Series and Random Single Amounts

\[ P_T = ? \]

\[ i = 10\% \]

\[ A = $5000 \]

\[ $2000 \]

Use \( P/A \) to get \( P_A \) in year 2:

\[ P_A = 5000(P/A,10\%,8) = 5000(5.3349) = $26,675 \]

Move \( P_A \) back to year 0 using \( P/F \):

\[ P_0 = 26,675(P/F,10\%,2) = 26,675(0.8264) = $22,044 \]

Move $2000 single amount back to year 0:

\[ P_{2000} = 2000(P/F,10\%,8) = 2000(0.4665) = $933 \]

Now, add \( P_0 \) and \( P_{2000} \) to get \( P_T \):

\[ P_T = 22,044 + 933 = $22,977 \]
Example Worked a Different Way
(Using F/A instead of P/A for uniform series)

The same re-numbered diagram from the previous slide is used

\[ P_T = ? \]

\[ F_A = ? \]

\[ A = \$5000 \]

\[ i = 10\% \]

\[ \$2000 \]

Solution:
Use F/A to get \( F_A \) in actual year 10:
\[ F_A = 5000(F/A,10\%,8) = 5000(11.4359) = \$57,180 \]

Move \( F_A \) back to year 0 using P/F:
\[ P_0 = 57,180(P/F,10\%,10) = 57,180(0.3855) = \$22,043 \]

Move \$2000 single amount back to year 0:
\[ P_{2000} = 2000(P/F,10\%,8) = 2000(0.4665) = \$933 \]

Now, add two P values to get \( P_T \):
\[ P_T = 22,043 + 933 = \$22,976 \]

Same as before

As shown, there are usually multiple ways to work equivalency problems
Example: Series and Random Amounts

Convert the cash flows shown below (black arrows) into an equivalent annual worth $A$ in years 1 through 8 (red arrows) at $i = 10\%$ per year.

\begin{align*}
0 & \quad 1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 & \quad 7 & \quad 8 \\
\text{A = \$3000} & & & & & & \hspace{2cm} \text{i = 10\%} \\
\text{A = \$1000} & & & & & & \\
\end{align*}

Approaches:

1. Convert all cash flows into $P$ in year 0 and use $A/P$ with $n = 8$
2. Find $F$ in year 8 and use $A/F$ with $n = 8$

Solution:

Solve for $F$: 
\begin{align*}
F &= 3000(F/A,10\%,5) + 1000(F/P,10\%,1) \\
&= 3000(6.1051) + 1000(1.1000) \\
&= \$19,415 \\

\text{Find A: } \\
A &= 19,415(A/F,10\%,8) \\
&= 19,415(0.08744) \\
&= \$1698
\end{align*}
Example 19
Shifted Arithmetic Gradients

- Shifted gradient begins at a time other than between periods 1 and 2
- Present worth $P_G$ is located 2 periods before gradient starts
- Must use multiple factors to find $P_T$ in actual year 0
- To find equivalent A series, find $P_T$ at actual time 0 and apply $(A/P,i,n)$
Example: Shifted Arithmetic Gradient

John Deere expects the cost of a tractor part to increase by $5 per year beginning 4 years from now. If the cost in years 1-3 is $60, determine the present worth in year 0 of the cost through year 10 at an interest rate of 12% per year.

\[ P_T = ? \]

\[ i = 12\% \]

\[ \begin{array}{c|c|c|c|c|c|c|c|c|c|c|c} 
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline 
\hline 
P_T & 60 & 60 & 60 & 65 & 70 & & & & & \\
\hline 
G = 5 & & & & & & & & & & \\
\hline 
\end{array} \]

Solution: First find \( P_2 \) for \( G = 5 \) and base amount ($60) in actual year 2

\[ P_2 = 60(P/A,12\%,8) + 5(P/G,12\%,8) = $370.41 \]

Next, move \( P_2 \) back to year 0

\[ P_0 = P_2(P/F,12\%,2) = $295.29 \]

Next, find \( P_A \) for the $60 amounts of years 1 and 2

\[ P_A = 60(P/A,12\%,2) = $101.41 \]

Finally, add \( P_0 \) and \( P_A \) to get \( P_T \) in year 0

\[ P_T = P_0 + P_A = $396.70 \]
Example 20
Shifted Geometric Gradients

Shifted gradient begins at a time other than between periods 1 and 2

Equation yields $P_g$ for all cash flows (base amount $A_1$ is included)

Equation ($i \neq g$):

$$P_g = A_1 \{1 - \left[\frac{(1+g)}{(1+i)}\right]^n/(i-g)\}$$

For negative gradient, change signs on both $g$ values

There are no tables for geometric gradient factors
Weirton Steel signed a 5-year contract to purchase water treatment chemicals from a local distributor for $7000 per year. When the contract ends, the cost of the chemicals is expected to increase by 12% per year for the next 8 years. If an initial investment in storage tanks is $35,000, determine the equivalent present worth in year 0 of all of the cash flows at $i = 15\%$ per year.
Example: Shifted Geometric Gradient

Gradient starts between actual years 5 and 6; these are gradient years 1 and 2.  

$P_g$ is located in gradient year 0, which is actual year 4

\[
P_g = 7000\{1-\left[\frac{1+0.12}{1+0.15}\right]^9/\left(0.15-0.12\right)\} = $49,401
\]

Move $P_g$ and other cash flows to year 0 to calculate $P_T$

\[
P_T = 35,000 + 7000(P/A,15\%,4) + 49,401(P/F,15\%,4) = $83,232
\]
Negative Shifted Gradients

For negative arithmetic gradients, change sign on G term from + to -

General equation for determining \( P \): \( P = \text{present worth of base amount} \ - P_G \)

For negative geometric gradients, change signs on both \( g \) values

\[
P_g = A_t \{1 - \left[\frac{(1-g)}{(1+i)}\right]^n/(i+g)\}
\]

All other procedures are the same as for positive gradients
Example: Negative Shifted Arithmetic Gradient

For the cash flows shown, find the future worth in year 7 at $i = 10\%$ per year

Solution:

Gradient $G$ first occurs between actual years 2 and 3; these are gradient years 1 and 2.

$P_G$ is located in gradient year 0 (actual year 1); base amount of $700$ is in gradient years 1-6.

$$P_G = 700(P/A,10\%,6) - 50(P/G,10\%,6) = 700(4.3553) - 50(9.6842) = 2565$$

$$F = P_G(F/P,10\%,6) = 2565(1.7716) = 4544$$
Summary of Important Points

P for shifted uniform series is *one period ahead* of first A; 
n is equal to number of A values

F for shifted uniform series is in *same period* as last A; 
n is equal to number of A values

For gradients, *first change* equal to G or g occurs 
between gradient years 1 and 2

For negative arithmetic gradients, change sign on G from + to -

For negative geometric gradients, change sign on g from + to -