# *AYDIN ADNAN MENDERES UNIVERSITY FACULTY OF ENGINEERING*

*Department of Electrical and Electronics Engineering*

**EE213 – Transform Techniques With Computer Applications**

**2022-2023, Fall**

**Presentation 7**

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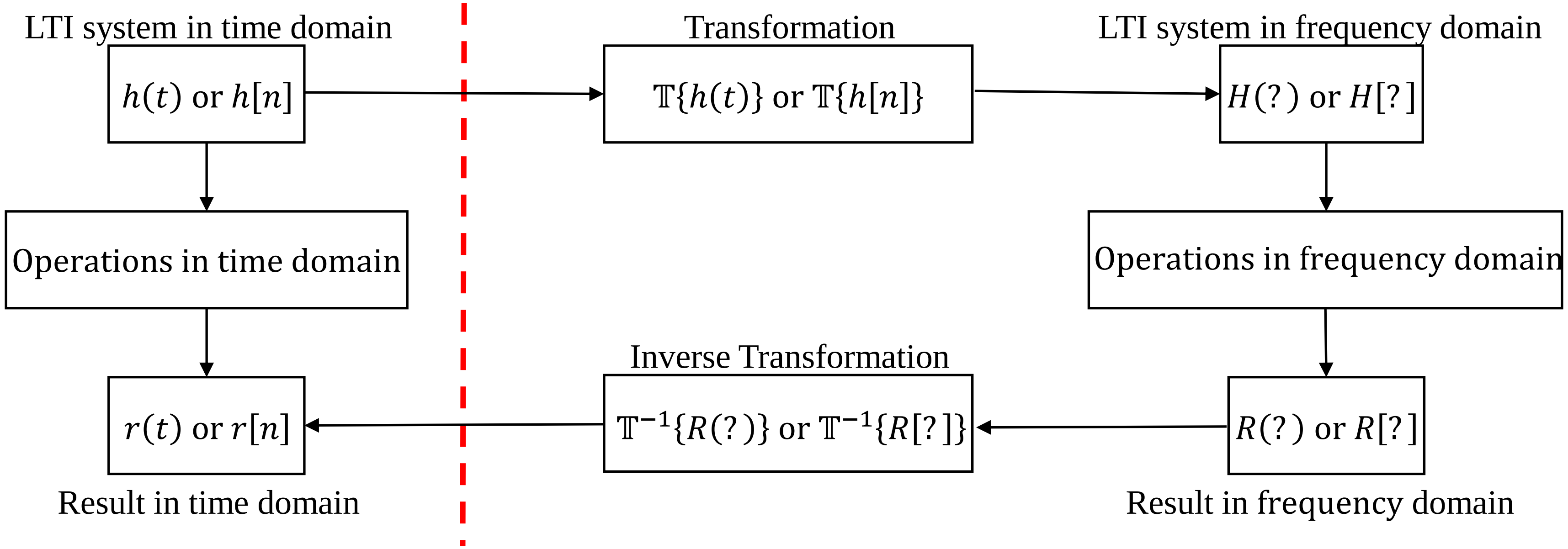
# Transformation Concept in Signals & Systems

- ✓ Transformations are used to eliminate the difficulties in processing linear time-invariant systems in the time domain.
- ✓ Thanks to the transformations, systems in the time domain (hence the signals in the time domain that make up these systems) are converted into the frequency domain.
- ✓ Thus, instead of time domain operations such as convolution, derivative, integral, which are more difficult and have high computational complexity, both time and computational complexity are saved by using the equivalents of these operations in the frequency domain.
- ✓ After the system and signals in the time domain are converted to the frequency domain and the necessary operations are performed in the frequency domain, the system and the obtained signals are converted back to the time domain by using inverse transformation.



# Laplace Transform

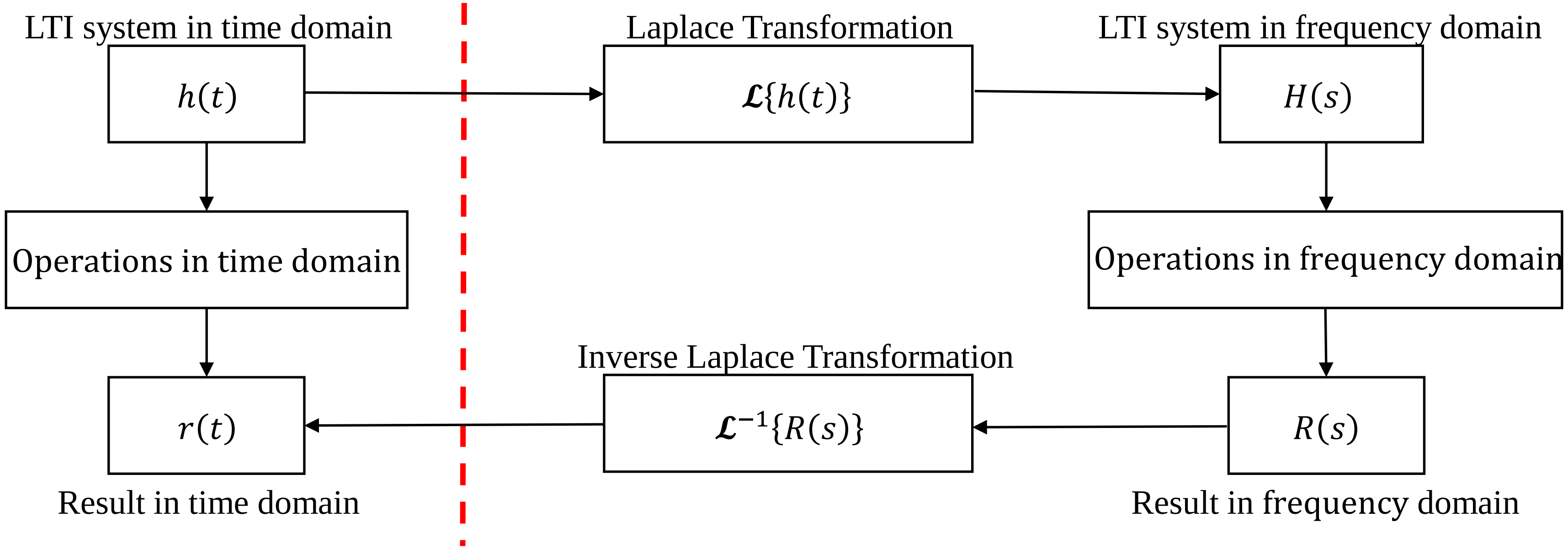
The Bilateral Laplace Transform of a signal $x(t)$ is defined as:

$$\mathcal{L}_b\{x(t)\} = X_b(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

If we define $x(t)$ to be 0 for $t < 0$ (for causal signals or systems), this gives us the Unilateral Laplace Transform:

$$\mathcal{L}\{x(t)\} = X(s) = \int_{0}^{\infty} x(t)e^{-st}dt$$

$\boldsymbol{s = \sigma + j\omega}$ **and** $\boldsymbol{\omega = 2\pi f}$



# Laplace Transform

**Ex:** Let's find the Unilateral Laplace Transform of unit step function.

$$\mathcal{L}\{u(t)\} = \int_0^{\infty} u(t)e^{-st}dt \ = \int_0^{\infty} e^{-st}dt = \frac{e^{-st}}{-s}\Big|_0^{\infty} = -\frac{1}{s}\left[\lim_{t\to\infty} e^{-st} - 1\right]$$

$$\lim_{t\to\infty} e^{-st} = \lim_{t\to\infty}\{e^{-\sigma t}e^{-j\omega t}\}$$

Remember from complex exponential function that; $\lim_{t\to\infty} e^{-st} = 0$ for $\sigma = Re(s) > 0$ and $\lim_{t\to\infty} e^{-st} = \infty$ for $\sigma = Re(s) < 0$

Hence Laplace Transform of $x(t) = u(t)$ exists and it is $X(s) = \frac{1}{s}$ if $Re(s) > 0$.

$Re(s) > 0$ is Region of Convergence (ROC) of this transform.

```
syms t;
u=heaviside(t);
laplace(u)
```

```
ans =
1/s
```

# Laplace Transform

**Ex:** Let's find the Unilateral Laplace Transform of $e^{-at}$.

$$\mathcal{L}\{e^{-at}\} = \int_0^{\infty} e^{-at}e^{-st}dt = \int_0^{\infty} e^{-(s+a)t}dt = -\frac{1}{s+a}e^{-(s+a)t}\Big|_0^{\infty} = -\frac{1}{s+a}\left[\lim_{t\to\infty} e^{-(s+a)t} - 1\right] = \frac{1}{s+a}$$

```
syms t a;                    ans =                              1
laplace(exp(-a*t))           1/(a + s)       pretty(ans)      -----
                                                              a + s
```

**Ex:** $\mathcal{L}\{\cos(at)\} = \int_0^{\infty} \cos(at)\, e^{-st}dt = \int_0^{\infty} \frac{1}{2}\left(e^{jat} + e^{-jat}\right)e^{-st}dt = \frac{1}{2}\left[\int_0^{\infty} e^{-(s-ja)t}dt + \int_0^{\infty} e^{-(s+ja)t}dt\right]$

$$= \frac{1}{2}\left(\frac{1}{s-ja} + \frac{1}{s+ja}\right) = \frac{s}{s^2+a^2}$$

```
syms t a;                 ans =                                   s
laplace(cos(a*t))         s/(a^2 + s^2)        pretty(ans)     -------
                                                                2    2
                                                               a  + s
```

# Inverse Laplace Transform

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}dt$$

The inverse Laplace transform expression (above expression) contains integration around a contour on the s-plane and is typically difficult to compute and **will not be applied in this course**.

However, one useful **alternative procedure** for obtaining a signal from **its Laplace transform is to expand X(s) into a partial fraction expansion**, and then to recognize the signal associated with each term in the expansion.

$$X(s) = \sum_{i=1}^{m} \frac{r_i}{s-p_i} \quad \Longleftrightarrow \quad x(t) = \sum_{i=1}^{m} e_i^{p_i t} u(t)$$

## Obtaining Partial Fraction Expansion of X(s) Using Matlab

```
[r,p,k] = residue(b,a)
```

finds the residues, poles, and direct term of a Partial Fraction Expansion of the ratio of two polynomials, where the expansion is of the form

$$\frac{b(s)}{a(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} = \frac{r_n}{s-p_n} + \ldots + \frac{r_2}{s-p_2} + \frac{r_1}{s-p_1} + k(s).$$

The inputs to residue are vectors of coefficients of the polynomials `b = [bm ... b1 b0]` and `a = [an ... a1 a0]`. The outputs are the residues `r = [rn ... r2 r1]`, the poles `p = [pn ... p2 p1]`, and the polynomial `k`.

# Inverse Laplace Transform

**Ex:** $X(s) = \dfrac{1}{s^2+3s+2}$ $\quad x(t) = ?$ $\quad X(s) = \dfrac{r_1}{s-p_1} + \dfrac{r_2}{s-p_2} = \dfrac{-1}{s+2} + \dfrac{1}{s+1}$

$$r_1 = (s+2)X(s)\Big|_{s=-2} = \frac{1}{s+1}\Big|_{s=-2} = -1 \qquad r_2 = (s+1)X(s)\Big|_{s=-1} = \frac{1}{s+2}\Big|_{s=-1} = 1$$

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2} \Longleftrightarrow x(t) = e^{-t}u(t) - e^{-2t}u(t)$$

```
syms s;
ilaplace(1/(s^2+3*s+2))

ans =
exp(-t) - exp(-2*t)
```

```
[r,p,k]=residue(1,[1 3 2])

r =
    -1
     1

p =
    -2
    -1

k =

    []
```

```
ilaplace(-1/(s+2)+1/(s+1))

ans =
exp(-t) - exp(-2*t)
```

# Some Properties of Laplace Transform

| Property Name | Transformation |
| --- | --- |
| **Linearity** | $ax_1(t) + bx_2(t) \Longleftrightarrow aX_1(s) + bX_2(s)$ |
| **Convolution** | $x_1(t) * x_2(t) \Longleftrightarrow X_1(s)X_2(s)$ |
| **First Order Derivative** | $\frac{dx(t)}{dt} \Longleftrightarrow sX(s) - x(0)$ |
| **$n^{th}$ Order Derivative** | $\frac{d^n x(t)}{dt^n} \Longleftrightarrow s^n X(s) - s^{n-1}x(0) - s^{n-2}\frac{dx(t)}{dt}\Big\vert_{t=0} - s^{n-3}\frac{d^2x(t)}{dt^2}\Big\vert_{t=0} - \cdots - s^0\frac{d^{n-1}x(t)}{dt^{n-1}}\Big\vert_{t=0}$ |
| **Integral** | $\int_0^t x(\tau)d\tau \Longleftrightarrow \frac{X(s)}{s}$ |
| **Time Shifting** | $x(t-a) \Longleftrightarrow e^{-as}X(s)$ |
| **Frequency Shifting** | $e^{at}x(t) \Longleftrightarrow X(s-a)$ |

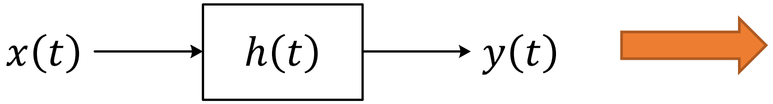
# Unilateral Laplace Transform Pairs for Common Signals

| Time Domain | Frequency Domain |
|---|---|
| $\delta(t)$ | $1$ |
| $u(t)$ | $\frac{1}{s}$ |
| $e^{-at}$ | $\frac{1}{s+a}$ |
| $t^n \quad (n = 1,2,\cdots)$ | $\frac{n!}{s^{n+1}}$ |
| $t^n e^{-at} \quad (n = 1,2,\cdots)$ | $\frac{n!}{(s+a)^{n+1}}$ |
| $\sin(\omega t)$ | $\frac{\omega}{s^2+\omega^2}$ |
| $\cos(\omega t)$ | $\frac{s}{s^2+\omega^2}$ |

# Analysis of LTI Systems Using Laplace Transform

The Laplace transform plays an important role in the analysis and representation of CT LTI systems.

From the convolution property, we have :




$$y(t) = x(t) * h(t)$$

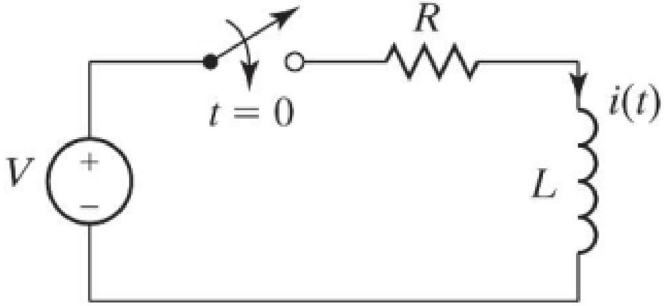
$$Y(s) = X(s).H(s)$$

$\mathcal{L}\{h(t)\} = H(s)$ is called the **system function** or **transfer function** of the LTI system.

# Practice 11:

In this practice, we will use Laplace transform to solve a simple circuit. First let's remember that, the Laplace transform of the derivative of a function, $f(t)$ is,

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$




Consider the $RL$ circuit, where $\mathrm{R} = 1\Omega, L = 1H$ and $V = 5V$.

The equation of voltages in the loop can be written as,

$$L\frac{di(t)}{dt} + Ri(t) = Vu(t)$$

a) Assuming that the initial value of current is zero, i.e. $i(0) = 0$, take the Laplace transform of the both sides of above Eq. and obtain $I(s)$. Then find the time domain equation for the current, $i(t)$.

b) Find the transfer function and impulse response of this circuit for the input $x(t) = Vu(t) = 5u(t)$ and the output $y(t) = i(t)$.

**Solution:**

a) $\frac{di(t)}{dt} + i(t) = 5u(t) \xrightarrow{\mathcal{L}} sI(s) - i(0) + I(s) = \frac{5}{s} \rightarrow I(s) = \frac{5}{s(s+1)} = \frac{5}{s^2+s} = \frac{r_1}{s+1} + \frac{r_2}{s}$

$r_1 = -5 \quad r_2 = 5 \rightarrow I(s) = \frac{-5}{s+1} + \frac{5}{s} \xrightarrow{\mathcal{L}^{-1}} i(t) = (5 - 5e^{-t})u(t)$

b) $\frac{di(t)}{dt} + i(t) = 5u(t) \rightarrow \frac{dy(t)}{dt} + y(t) = x(t) \xrightarrow{\mathcal{L}} sY(s) + Y(s) = X(s) \rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+1}$

$\downarrow \mathcal{L}^{-1}$

$h(t) = e^{-t}u(t)$

# Practice 12:

By using Matlab, first calculate the Laplace transform of the signal $x(t) = 5t + \sin(2t)$, then calculate the Laplace transform of the derivative of the same signal, $\frac{dx(t)}{dt}$, and show that $\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = sX(s)$.

# Practice 13:

Using Matlab, obtain the output signal $y(t)$ of the system, whose input-output relationship is given by the following differential equation, under initial conditions given as $y(0) = 1$ and $\left.\frac{dy(t)}{dt}\right|_{t=0} = y'(0) = -3$. Then, obtain the output signal $y(t)$ of the same system, this time under zero initial conditions, and plot the output signals you found for both cases on the same graph.

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 2y(t) = 5u(t)$$

**Solution:** We will take the Laplace transform of the both sides of above equation, but first let's remember the derivative property of Laplace transform and apply it to the derivative terms of the above equation:

$$\mathcal{L}\left\{\frac{d^ny(t)}{dt^n}\right\} = s^nY(s) - s^{n-1}y(0) - s^{n-2}\left.\frac{dy(t)}{dt}\right|_{t=0} - s^{n-3}\left.\frac{d^2y(t)}{dt^2}\right|_{t=0} - \cdots - s^0\left.\frac{d^{n-1}y(t)}{dt^{n-1}}\right|_{t=0}$$

$$\mathcal{L}\left\{\frac{d^2y(t)}{dt^2}\right\} = s^2Y(s) - sy(0) - s^0\left.\frac{dy(t)}{dt}\right|_{t=0} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s + 3$$

$$\mathcal{L}\left\{\frac{dy(t)}{dt}\right\} = sY(s) - s^0y(0) = sY(s) - 1$$

Now we can take the Laplace transform of the both sides of the system's differential equation:

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 2y(t) = 5u(t) \quad \xrightarrow{\mathcal{L}} \quad s^2Y(s) - s + 3 + 2(sY(s) - 1) + 2Y(s) = \frac{5}{s}$$

## Solution (Cont.):

$$s^2Y(s) - s + 3 + 2(sY(s) - 1) + 2Y(s) = \frac{5}{s}$$

$$s^3Y(s) - s^2 + 3s + 2s^2Y(s) - 2s + 2sY(s) = 5$$

$$Y(s)[s^3 + 2s^2 + 2s] = s^2 - 3s + 2s + 5$$

$$Y(s) = \frac{s^2 - s + 5}{s^3 + 2s^2 + 2s}$$

Output signal $y(t)$ for given initial conditions can be obtained by taking inverse Laplace transform in Matlab.

Output signal $y(t)$ for zero initial conditions can be obtained by taking the Laplace transform of the both sides of the system's differential equation:

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 2y(t) = 5u(t) \xrightarrow{\mathcal{L}} s^2Y(s) + 2sY(s) + 2Y(s) = \frac{5}{s}$$

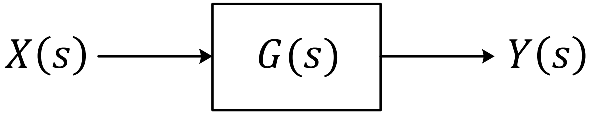
$$Y(s) = \frac{5}{s(s^2 + 2s + 2)}$$

Output signal $y(t)$ for zero initial conditions can be obtained by taking inverse Laplace transform in Matlab.

# Defining Systems Using Transfer Function in Matlab

A continuous-time transfer function is expressed as the ratio of polynomials N(s) and D(s), called the numerator and denominator polynomials, respectively.

$$G(s) = \frac{N(s)}{D(s)}$$




Linear systems can be represented as transfer functions in polynomial form:

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

$$G(s) = \frac{s^2 - 3s - 4}{s^2 + 5s + 6}$$

**Ex:** This example shows how to create below continuous-time transfer function from its numerator and denominator coefficients using **tf**.

$$G(s) = \frac{s}{s^2 + 3s + 2}$$

```
num = [1 0];
den = [1 3 2];
G = tf(num,den)
```

```
G =

        s
  -------------
  s^2 + 3 s + 2

Continuous-time transfer function.
```

**num** and **den** are the numerator and denominator polynomial coefficients in descending powers of $s$. For example, **den = [1 3 2]** represents the denominator polynomial $s^2 + 3s + 2$.

# Defining Systems Using Transfer Function in Matlab

Linear systems can also be represented as transfer functions in factorized (zero-pole-gain) form:

$$G(s) = k\frac{(s - z(1))(s - z(2))\ldots(s - z(m))}{(s - p(1))(s - p(2))\ldots(s - p(n))} \qquad G(s) = \frac{(s+1)(s-4)}{(s+2)(s+3)}$$

Here, $z$ and $p$ are the vectors of real-valued or complex-valued zeros and poles, and $k$ is the real-valued or complex-valued scalar gain.

<u>Ex:</u> This example shows how to create below continuous-time transfer function in factored form using **`zpk`**.

$$G(s) = 5\frac{s}{(s+1+i)(s+1-i)(s+2)}$$

```
Z = [0];
P = [-1-1i -1+1i -2];
K = 5;
G = zpk(Z,P,K)
```

```
G =

            5 s
  --------------------
  (s+2) (s^2 + 2s + 2)

Continuous-time zero/pole/gain model.
```

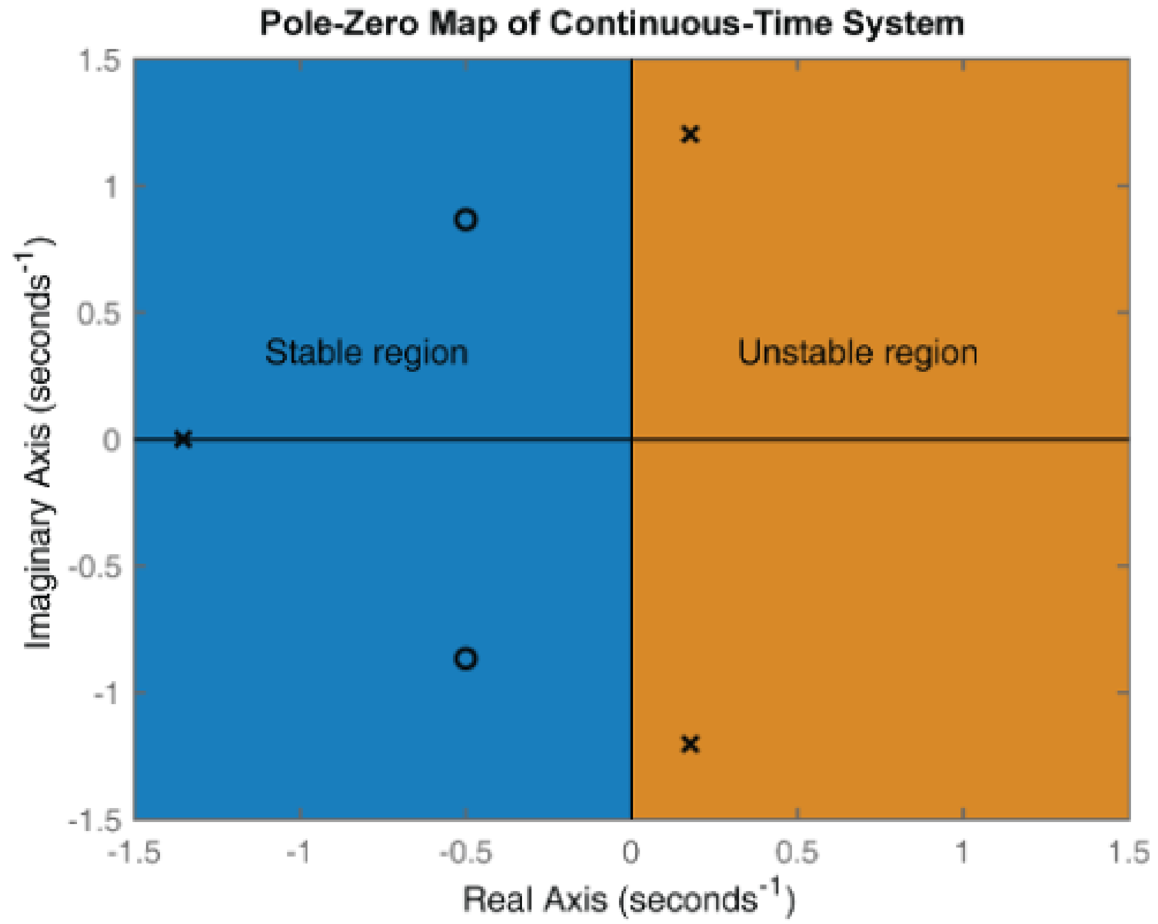
**`Z`** and **`P`** are the zeros and poles (the roots of the numerator and denominator, respectively). **`K`** is the gain of the factored form.

For example, $G(s)$ has a real pole at $s = -2$ and a pair of complex poles at $s = -1 \pm i$.

The vector **`P = [-1-1i -1+1i -2]`** specifies these pole locations.

# Defining Systems Using Transfer Function in Matlab

Matlab function **`pzmap(sys)`** creates a pole-zero plot of the continuous time dynamic system model **`sys`**.
x and o indicates the poles and zeros respectively, as shown in the following figure.

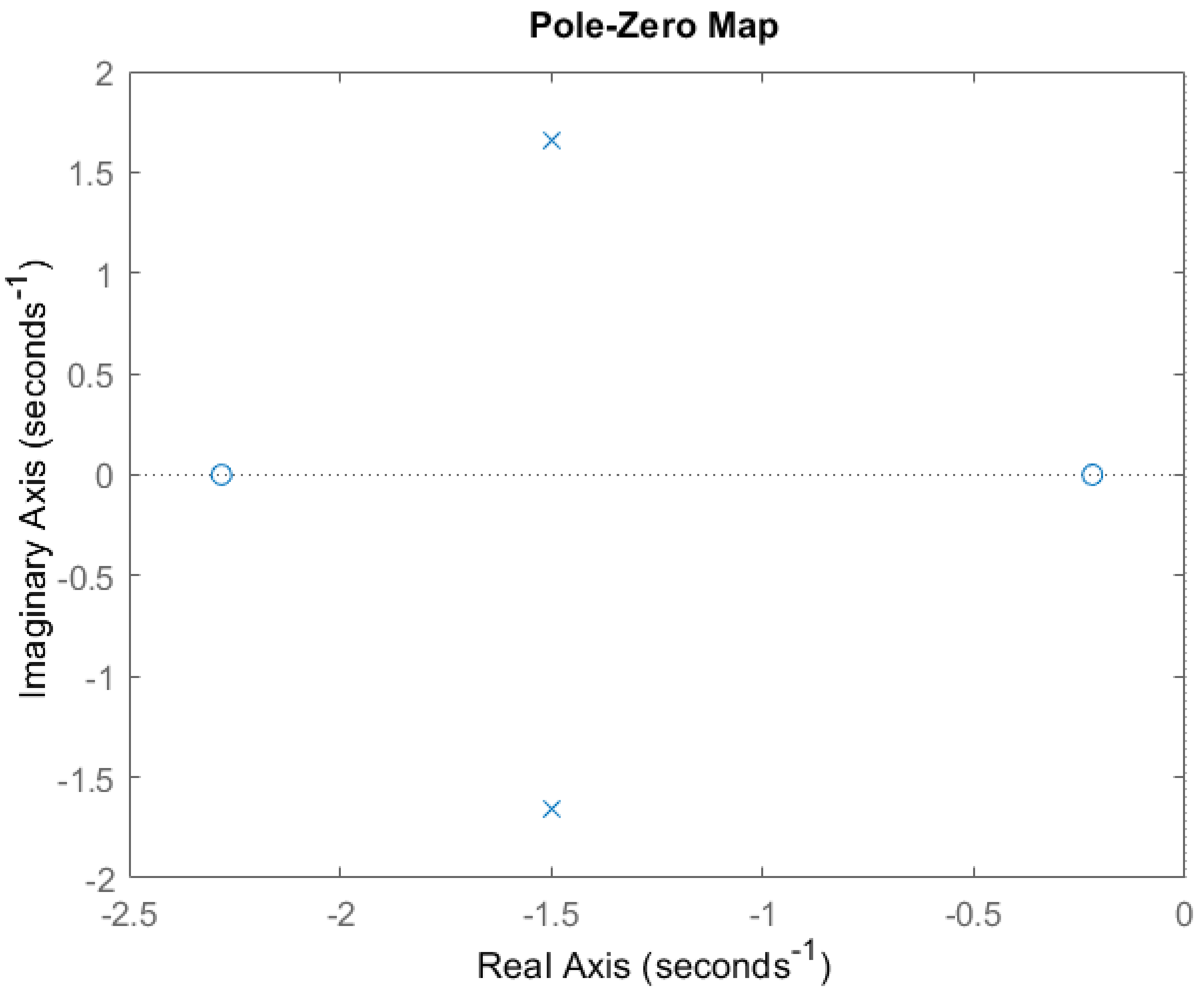


# Defining Systems Using Transfer Function in Matlab

**Ex:** Plot and calculate the poles and zeros of the continuous-time system represented by the following transfer function:

$$H(s) = \frac{2s^2 + 5s + 1}{s^2 + 3s + 5}$$

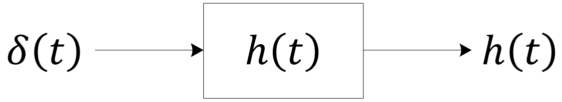
```
H = tf([2 5 1],[1 3 5]);
pzmap(H)


[p,z] = pzmap(H)

p =

  -1.5000 + 1.6583i
  -1.5000 - 1.6583i


z =

   -2.2808
   -0.2192
```



# Defining Systems Using Transfer Function in Matlab

Matlab function **`impulse`** calculates the unit impulse response of a dynamic system model.

For continuous-time dynamic systems, the impulse response, $h(t)$ is the response to a Dirac input $\delta(t)$.



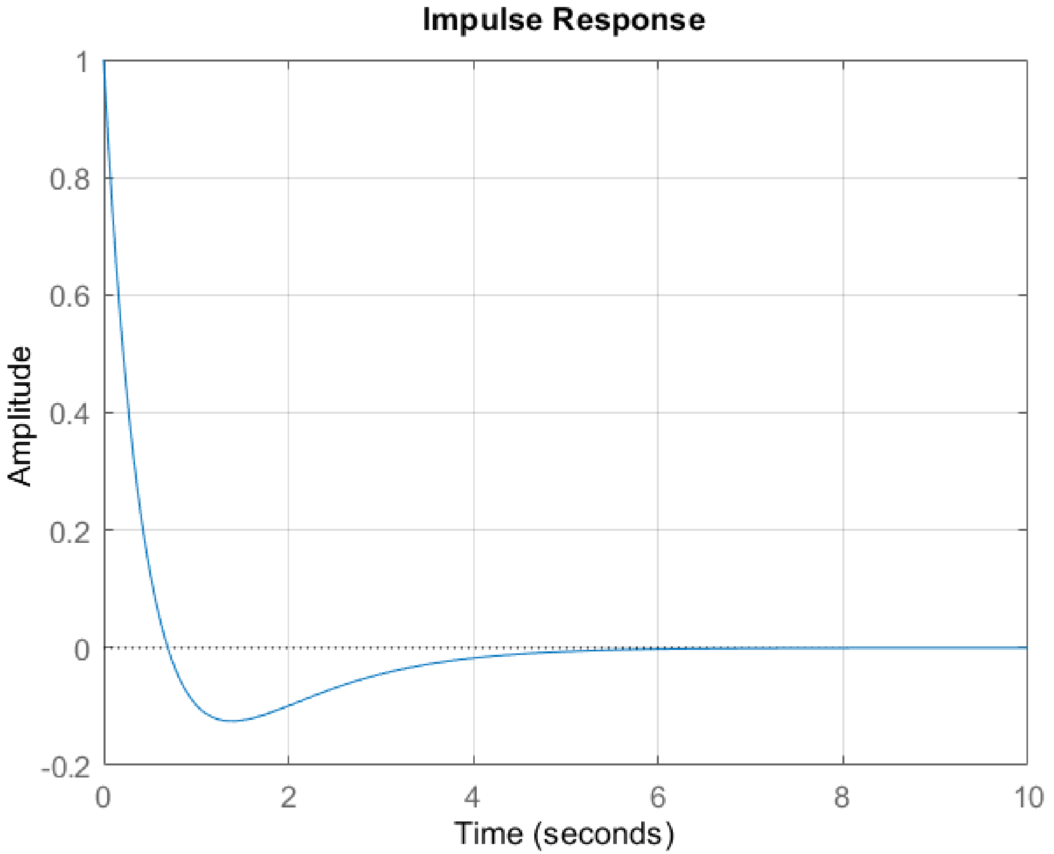

**`impulse(sys)`** plots the impulse response of the dynamic system model **`sys`**.

**`[y,t] = impulse(sys)`** returns the output impulse response y and the time vector t. No plot is drawn on the screen.

**Ex:** First create the continuous-time system of which transfer function is given below, and then plot the impulse response of this system.

$$G(s) = \frac{s}{s^2 + 3s + 2}$$

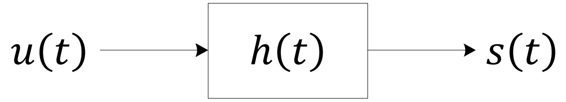
```
num = [1 0];
den = [1 3 2];
G = tf(num,den);
impulse(G)
grid on
```



# Defining Systems Using Transfer Function in Matlab

Matlab function **step** calculates the response of a dynamic system model to a step input of unit amplitude.

For continuous-time dynamic systems, the step response, $s(t)$ is the response to a unit step input $u(t)$.



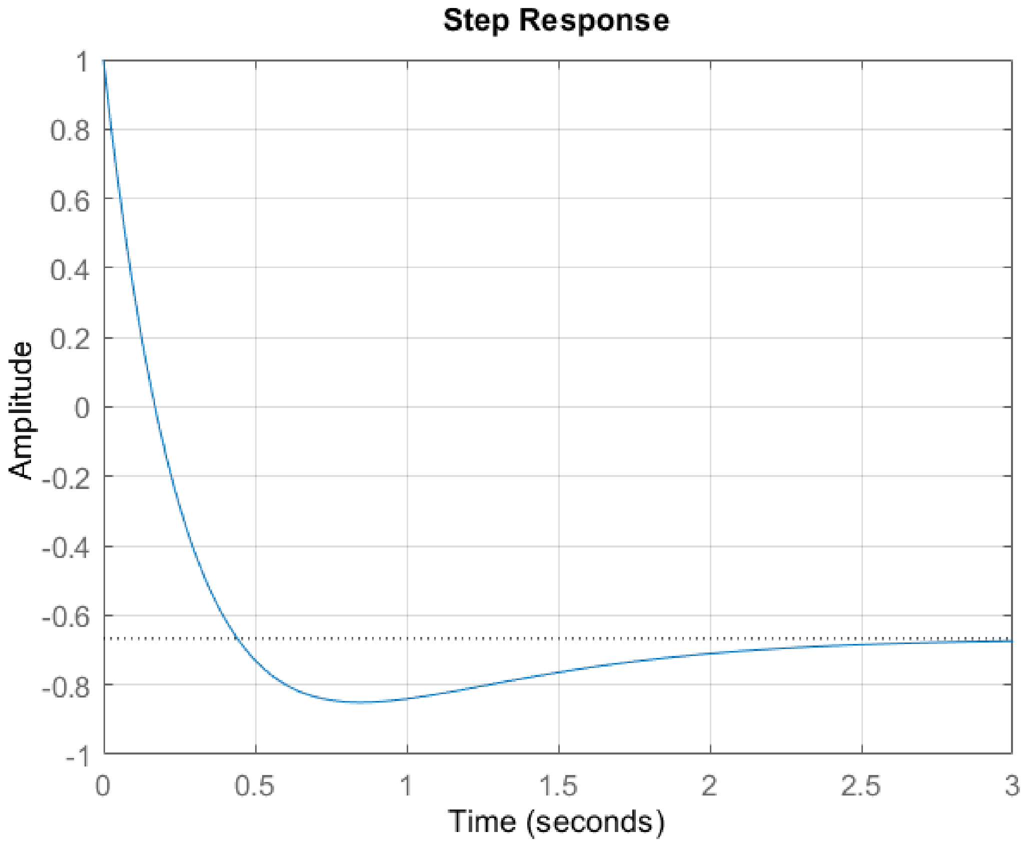

**step(sys)** plots the step response of the dynamic system model **sys**.

**[y,t] = step(sys)** returns the output step response y and the time vector t. No plot is drawn on the screen.

**Ex:** First create the continuous-time system of which transfer function is given below, and then plot the step response of this system.

$$G(s) = \frac{(s+1)(s-4)}{(s+2)(s+3)}$$

```
Z = [-1 4];
P = [-2 -3];
K=1;
G = zpk(Z,P,K);
step(G)
grid on
```



# END OF PRESENTATION 7

## References:

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✓Signals and Systems, Alan V.  Oppenheim, Alan S. Willsky, and S. Hamid Nawab, 2nd Edition, Pearson New International Edition, 2014.

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